



INVARIANT NORMALIZATION OF NON-AUTONOMOUS HAMILTONIAN SYSTEMS†

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A new algorithm is proposed for reducing non-autonomous Hamiltonian systems to normal Birkhoff form. The criterion for the normal form is the condition that the vector fields of the perturbed and unperturbed parts of the system should commute. The invariant character of the criterion enables the system to be normalized in a unified way, without first simplifying the unperturbed part and without distinguishing between resonance and non-resonance, or autonomous and non-autonomous, cases. The whole algorithm reduces to a one-dimensional recurrence formula. The result is obtained by using the Campbell–Hausdorff formula for the ring of asymptotic forms, as well as the solution of a homological equation in the form of a quadrature. Three examples are considered to illustrate the various special features of the new algorithm. One of the examples is of interest for nuclear magnetic resonance theory. © 2002 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The method of local analysis of systems of ordinary differential equations is based on representing the system as a sum of two parts with different meanings. The first part is called the *degenerate system* (more frequently the generating or unperturbed system). The second part is called the *perturbation*. An arbitrary small perturbation may lead to considerable changes in the behaviour of the system, even a qualitative change in the topology of its phase flow. This not only explains the term “degenerate system,” but also makes local analysis meaningful, since otherwise there would be little point in taking small terms into account in the equations.

One of the main methods of local analysis is the method of the Poincaré normal form. If one is concerned with a Hamiltonian system, its Poincaré normal form is related to the normal form of the Hamiltonian, which is known as the Birkhoff normal form [1]. The most compact definition of this form can be found in [2].

We will illustrate the nature of the definition using the example of the Hamiltonian of an autonomous conservative oscillatory system, represented in the neighbourhood of an equilibrium position in the form

$$H(x, y) = i \sum_{k=1}^n \lambda_k x_k y_k + \sum_{m, l > 0} h_{ml} x^m y^l \quad (1.1)$$

where x and y are complex combinations of coordinates and momenta, $x = p + iq$, $y = p - iq$, the positive constants λ_k have the meaning of frequencies of natural oscillations of the linear part of the system, whose Hamiltonian is written in diagonal form, and we have used the abbreviated notation $x^m = x_1^{m_1} \dots x_n^{m_n}$ in representing the Hamiltonian of the non-linear part. In this case we say that Hamiltonian (1.1) has the Birkhoff normal form if the non-linear part contains only terms for which

$$\sum_k (m_k - l_k) \lambda_k = 0 \quad (1.2)$$

Such terms are known as resonance terms. The definition in the non-autonomous case is more complicated.

If the Hamiltonian of the non-linear part of the system does not have this form, it may be brought to the appropriate form by using canonical polynomial changes of variables and choosing the coefficients in such a way that non-resonance terms vanish.

There are two known methods of constructing such changes of variables: one based on using generating functions (this was Birkhoff’s approach), and another in which the generating functions are replaced by Lie generators [3]. The second method is more convenient, since it does not require the inversion of power series, which is necessary in the case of generating functions.

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The method of Lie generators is sometimes referred to as Hori's method. However, it should be noted that Hori himself used the method to look for an additional first integral [4], rather than to normalize Hamiltonians. The use of Lie generators to form recurrence normalization procedures was implemented in [5].†

2. INVARIANT DEFINITION OF THE BIRKHOFF NORMAL FORM

The definition of the normal form outlined above, based on using complex combinations of generalized coordinates and generalized momenta and reducing the unperturbed part to diagonal form, compares favourably with other definitions known from the literature (such as those making direct use of canonical variables or of action-angle variables). This definition does not involve a separate description of the resonance and non-resonance cases, thus achieving greater compactness.

The definition may be made even more compact if it does not depend at all on the specific variables in terms of which the original Hamiltonian is written. To that end, we need only appeal to a well-known general property which characterizes the normal form irrespective of the way it is defined: the terms in representation (1.1) satisfying condition (1.2) and called "resonance terms" cannot be eliminated by any canonical transformation; these terms will be first integrals of the unperturbed system.

It is obvious that this property does not depend on the specific variables in terms of which the system is described; it is a fundamental topological property of the normal form, indicating that the vector fields of the perturbed and unperturbed systems commute. It is this property that is most naturally taken as the definition of the normal form. The actual form of the resonance terms for a specific choice of variables will then be a corollary of the definition.

Definition. The perturbed Hamiltonian

$$H(t, q, p) = H_0(t, q, p) + H_*(t, q, p) \quad (2.1)$$

has the Birkhoff normal form if and only if the perturbation is a first integral of the unperturbed part:

$$\partial H_* / \partial t + \{H_0, H_*\} = 0 \quad (2.2)$$

In what follows, with the aim of reducing an arbitrary perturbed Hamiltonian to such a form, we will additionally require that the Hamiltonian be a periodic function of time and that the general solution of the unperturbed system be known and conditionally periodic.

The above definition may be made even more compact if we assume the time to be a new generalized coordinate: $q_0 \equiv \lambda_0 t$ (where λ_0 is the quotient of 2π by the period) and introduce the momentum p_0 conjugate to this coordinate, so that the Hamiltonian may be rewritten as $H \rightarrow \lambda_0 p_0 + H$. Then the definition of the normal form as $\{H_0, H_*\} = 0$ will be common to both the autonomous and non-autonomous cases.

The above definition will be convenient in what follows in formulating the normalization algorithm. Like the definition itself, the algorithm will be invariant to the variables used.

3. THE NORMALIZATION ALGORITHM

We shall seek a canonical transformation which reduces the Hamiltonian

$$H(q, p) = H_0(q, p) + H_*(q, p)$$

to normal form with the aid of Lie generators.

A Lie generator is defined as the operator of a certain auxiliary Hamiltonian system whose phase flow induces a single-parameter family of canonical transformations, defined in the same phase space in which the Hamiltonian to be transformed is defined. The Hamiltonian of this auxiliary system is called the generating Hamiltonian.

Let $G(q, p)$ be the required generating Hamiltonian. This means that the solution of the Cauchy problem

$$dp/d\tau = \partial G/\partial p, \quad dp/d\tau = -\partial G/\partial q; \quad q(0) = u, \quad p(0) = v$$

†See also: MARKEYEV, A. P. and SOKOLOVSKII, A. G., Some computational algorithms for the normalization of Hamiltonian systems. Preprint No. 31. Inst. Problem Mekhaniki Akad. Nauk SSSR, Moscow, 1976.

defines the required canonical transformations

$$q = q(\tau, u, v), \quad p = p(\tau, u, v)$$

These transformations and their inverses may be written in terms of the following Lie series

$$q = u + \tau\{u, G\} + \frac{\tau^2}{2!} \{\{u, G\}, G\} + \dots, \quad p = v + \tau\{v, G\} + \frac{\tau^2}{2!} \{\{v, G\}, G\} + \dots,$$

$$u = q - \tau\{q, G\} + \frac{\tau^2}{2!} \{\{q, G\}, G\} - \dots, \quad v = p - \tau\{p, G\} + \frac{\tau^2}{2!} \{\{p, G\}, G\} - \dots$$

The arguments q and p in the generating Hamiltonian $G(q, p)$ in the formulae for the transformations themselves have been replaced by u and v : $G(u, v)$. The function G itself is the same in both cases.

The transformed Hamiltonian is also defined in terms of the initial Hamiltonian by a Lie series

$$\bar{H}(u, v) = H(u, v) + \tau\{H, G\} + \frac{\tau^2}{2!} \{\{H, G\}, G\} + \dots \tag{3.1}$$

In what follows we shall assume that the smallness of the perturbation is formalized by the presence of a small parameter ϵ as a factor before it

$$H(q, p) = H_0(q, p) + \epsilon H_*(q, p)$$

In real problems the small parameter may appear in a natural way as an actual parameter which is indeed small in the sense of the problem being solved. If there is no such parameter, it may be introduced by means of a similarity transformation: $q \rightarrow \epsilon q, p \rightarrow \epsilon q$, where ϵ is a scalar parameter considered in the neighbourhood of zero.

Let us assume that the generating Hamiltonian is being sought as a function of ϵ : $G(q, p, \epsilon)$. We define an *asymptotic form of the k th order* of $G(p, q, \epsilon)$ to be any function $G_k(p, q, \epsilon)$ that differs from the exact one by small terms of higher order than ϵ^k : $G_k = G + o(\epsilon^k)$. An analogous definition is borne in mind for the required Hamiltonian in normal form, as well as for the initial Hamiltonian if the small parameter was introduced by way of a similarity transformation.

We recall the basic properties of the ring of asymptotic forms:

- 1) addition is defined by $(G' + G'')_k = G'_k + G''_k$; the zero of the ring of k -th order asymptotic forms is any function of order greater than k : $0_k = o(\epsilon^k)$;
- 2) multiplication is defined by $(G' \cdot G'')_k = G'_k \cdot G''_k$; the unit is $1_k = 1 + o(\epsilon^k)$ (the ring of asymptotic forms is unitary);
- 3) displacement along the scale of orders: $(\epsilon^s \cdot G)_k = \epsilon^s G_{k-s}$ ($k - s \geq 0$).

In what follows we formulate the problem of successively determining the asymptotic forms of the unknown Hamiltonians \bar{H} and G , beginning with the lowest-order ones. To that end we rewrite formula (3.1) for the successive asymptotic forms, identifying the parameter of the generating Hamiltonian group τ with the small parameter ϵ : $\tau = \epsilon$. As a result we obtain a chain of finite series

$$\bar{H}_0(u, v) = H_0(u, v), \quad \bar{H}_1(u, v) = H_1(u, v) + \tau\{H_0, G_0\}$$

$$\bar{H}_2(u, v) = H_2(u, v) + \tau\{H_1, G_1\} + \frac{\tau^2}{2!} \{\{H_0, G_0\}, G_0\}, \dots$$

$$\dots, \bar{H}_k(u, v) = H_k(u, v) + \tau\{H_{k-1}, G_{k-1}\} + \dots + \frac{\tau^k}{k!} \underbrace{\{\dots\{H_0, G_0\}, G_0\}, \dots}_{k\text{-times}}, \dots$$

The aforementioned properties of the ring of asymptotic forms have been used.

We again use the property of the displacement of order to transform the following bracket

$$\{H_{k-1}, G_{k-1}\} = \{H_0, G_{k-1}\} + \{H_{k-1} - H_0, G_{k-1}\} = \{H_0, G_{k-1}\} + \{H_{k-1} - H_0, G_{k-2}\}$$

The last equality follows from the fact that $H_{k-1} - H_0 \approx \epsilon$. Taking this representation into account, we rewrite (3.2) as

$$\bar{H}_0 = H_0, \quad \bar{H}_1 = \tau\{H_0, G_0\} + H_1 \tag{3.3}$$

$$\begin{aligned} \bar{H}_2 &= \tau\{H_0, G_1\} + \tau\{H_1 - H_0, G_0\} + \frac{\tau^2}{2} \{\{H_0, G_0\}, G_0\} + H_2, \dots \\ \dots, \bar{H}_k &= \tau\{H_0, G_{k-1}\} + R_k \quad (k > 1), \dots \end{aligned}$$

where R_k denotes a function containing only asymptotic forms of order lower than G_{k-1}

$$R_k = H_k + \tau\{H_{k-1} - H_0, G_{k-2}\} + \sum_{i=2}^k \frac{\tau^i}{i!} \underbrace{\{\dots\{H_{k-i}, G_{k-i}\}, G_{k-i}\}, \dots} \tag{3.4}$$

The first approximation of our method for reducing the Hamiltonian H to normal form involves finding \bar{H}_1 and G_0 from the second equation of system (3.3). If attention is being confined to this approximation, the asymptotic form G_0 of the generating Hamiltonian need not be constructed, since it is needed to find the second approximation.

The complete construction of the first approximation proceeds as follows. Together with the equation of the first approximation, we consider the normal form criterion

$$\bar{H}_1 = \tau\{H_0, G_0\} + H_1, \quad \{H_0, \bar{H}_1\} = 0 \tag{3.5}$$

This is a system of two first-order linear partial differential equations for the two functions $\bar{H}_1(u, v)$ and $G_0(u, v)$. Systems of this type are known as *homological* [6].

We shall present the solution of the homological system in explicit form. To do so we write the general solution

$$q = q(t, u, v), \quad p = p(t, u, v) \tag{3.6}$$

of the generating system

$$dq/dt = \partial H_0 / \partial p, \quad dp/dt = -\partial H_0 / \partial q$$

with initial conditions $q(0) = u, p(0) = v$.

If solution (3.6) is substituted into system (3.5) (replacing the variable u by $q(t, u, v)$ and the variable v by $p(t, u, v)$), the second equation becomes an identity. In the first equation, \bar{H}_1 remains unchanged, since \bar{H}_1 is by definition a first integral of the generating system. The Poisson bracket along trajectories of the generating system is equal to the total derivative of G_0 with respect to time

$$\{H_0, G_0\} = dG_0 / dt$$

Therefore, integrating both sides of this equation with respect to time, we obtain

$$\int_0^t H_1[q(t, u, v), p(t, u, v)] dt = t\bar{H}_1 + \tau[G_0(u, v) - G_0(q(t, u, v), p(t, u, v))] \tag{3.7}$$

We have thus shown that integration of the first-order asymptotic form of the given Hamiltonian with respect to time along trajectories of the generating system yields the first-order asymptotic form of the normal form as the coefficient of t , and the first-order asymptotic form of the generating Hamiltonian as the coefficient of τ , which does not depend on time.

We now proceed to the second approximation. Integrating the third equation of system (3.3) with respect to time along trajectories of (3.6), we obtain

$$\int_0^t (\tau\{H_1 - H_0, G_0\} + \frac{\tau^2}{2} \{\{H_0, G_0\}, G_0\} + H_2) dt = t\bar{H}_2 + \tau G_1 - \tau G_1(t)$$

and so on.

An arbitrary approximation is defined by the following recurrence formula

$$\int_0^t R_k[q(t, u, v), p(t, u, v)] dt = t\bar{H}_k + \tau G_{k-1}(u, v) - \tau G_{k-1}[q(t, u, v), p(t, u, v)] \tag{3.8}$$

The function $R_k(u, v)$ is given by formula (3.4) and is uniquely defined by the previous approximation.

Thus, in any approximation, the coefficient of t on the right of equality (3.8) is the required asymptotic form of the normal form, and the coefficient of τ , which is independent of time, is the corresponding asymptotic form of the generating Hamiltonian.

Remark. The function $G_{k-1}(u, v)$ in formula (3.8) may also be constructed by direct computation, as follows. As established, the asymptotic form of the normal form \bar{H}_k is the mean value of the function R_k over time along solutions of the generating system: $\bar{H}_k = \langle R_k \rangle$. Therefore, in order to find the asymptotic form of the generating Hamiltonian G_{k-1} , we need only solve the equation $\tau\{H_0, G_{k-1}\} + \bar{R}_k = 0$, where $\bar{R}_k = R_k - \langle R_k \rangle$ is the increment to the mean. This equation may be solved as follows (see, e.g., [7]). Substitute the solutions of the generating system $\dot{u} = \partial H_0 / \partial v$, $\dot{v} = -\partial H_0 / \partial u$ in the variables u and v , that is, $u = \varphi(t, u_0, v_0)$ and $v = \psi(t, u_0, v_0)$, into the expression for $\bar{R}_k(u, v)$ and calculate

$$N_k(t, u_0, v_0) = \int \bar{R}_k[\varphi(t, u_0, v_0), \psi(t, u_0, v_0)] dt.$$

After that, solving for u_0 and v_0 in terms of u and v , find

$$G_{k-1}(u, v) = -(1/\tau)N_k[t, \varphi(-t, u, v), \psi(-t, u, v)]$$

It is easily shown that the function thus obtained is indeed not explicitly dependent on time.

4. SPECIAL FEATURES OF THE CASE OF A NON-AUTONOMOUS HAMILTONIAN

In the non-autonomous case, the Hamiltonian has the form (2.1) and, as already indicated in Section 2, the algorithm may be applied after adding an impulse adjoint to the time t , denoted by T , to the Hamiltonian

$$H(t, T, q, p) = T + H_0(t, q, p) + H_*(t, q, p) \quad (4.1)$$

For convenient application of the formulae obtained in Section 3 to specific examples, it is better to denote the variables t and T by q_0 and p_0 .

Let us assume that Hamiltonian (4.1) has already been reduced to normal form, that is, Eq. (2.2) is true for (2.1). A non-autonomous Hamiltonian in normal form is also non-autonomous. The same may be said of the corresponding Hamiltonian system. However, the system has an important property, which considerably simplifies its analysis compared to non-autonomous systems of general form.

Theorem. If a system with Hamiltonian (2.1) satisfies the normal form condition, then, in order to construct the general solution of the corresponding Hamilton equations $\dot{q} = H_p, \dot{p} = -H_q$, it is sufficient:

a) to find the general solution $q = q(t', u', v'), p = p(t', u', v')$ of the generating system

$$\dot{q} = \partial H_0 / \partial p, \quad \dot{p} = -\partial H_0 / \partial q$$

b) to find the general solution $q = q(t'', u'', v''), p = p(t'', u'', v'')$ of the system defined only by the perturbation

$$\dot{q} = \partial H_* / \partial p, \quad \dot{p} = -\partial H_* / \partial q$$

provided that the time occurring explicitly in the Hamiltonian in this system is equated to zero: $H_*(0, p, q)$.

The general solution of the initial non-autonomous system will then be represented by the composition in any order of the solutions thus obtained (instead of the arbitrary constants in the solution of the second system one substitutes the solutions of the first, and vice versa).

Proof. We shall use the extension of the Hamiltonian to the autonomous form (4.1) and write the Hamiltonian equations as

$$i = 1, \quad \dot{T} = -\frac{\partial H_0}{\partial t} - \frac{\partial H_*}{\partial p}, \quad \dot{q} = \frac{\partial H_0}{\partial p} + \frac{\partial H_*}{\partial p}, \quad \dot{p} = -\frac{\partial H_0}{\partial q} - \frac{\partial H_*}{\partial q}$$

with initial condition $t(0) = 0$.

By the condition $\{T + H_0, H_*\} = 0$, the general solution of this system is the composition of solutions of the following systems (a proof may be found, e.g. in [7])

$$i = 1, \quad \dot{T} = -\frac{\partial H_0}{\partial t}, \quad \dot{q} = \frac{\partial H_0}{\partial p}, \quad \dot{p} = -\frac{\partial H_0}{\partial q}, \quad t(0) = 0 \quad (\text{system 1})$$

$$i = 0, \quad \dot{T} = -\frac{\partial H_*}{\partial t}, \quad \dot{q} = \frac{\partial H_*}{\partial p}, \quad \dot{p} = -\frac{\partial H_*}{\partial q}, \quad t(0) = 0 \quad (\text{system 2})$$

Integrating the first equations in both systems, substituting the results thus found, $t \equiv t$ for the first system and $t \equiv 0$ for the second, into the other equations, and carrying out the required composition after their solution, one obtains the property stipulated in the theorem.

5. EXAMPLES

The following examples will illustrate various features of the proposed normalization algorithm.

In the first example, which is familiar from the textbook literature, the technique of constructing two approximations is illustrated. It easily demonstrates the difference between this technique and the technique used in the known algorithms.

Example 1. The Duffing equation

$$\ddot{q} + q + q^3 = 0$$

By making the substitution $x = q - ip, y = q + ip$, the Hamiltonian corresponding to this equation may be reduced to the form

$$H = i(xy + \varepsilon(x + y)^4 / 32), \quad \varepsilon = 1$$

First approximation of the normal form. The trajectories of the generating system, whose Hamiltonian is $H_0 = ixy$, are $x = ue^{it}, y = ve^{-it}$. Integrating the Hamiltonian along them, we find

$$\begin{aligned} \int_0^t H dt &= i \int_0^t \left[uv + \frac{\tau}{32} (ue^{it} + ve^{-it})^4 \right] dt = \\ &= iuv + \frac{\tau}{32} \left(\frac{1}{4} u^4 e^{4it} + 2u^3 v e^{2it} + 6iuv^2 v^2 - 2u v^3 e^{-2it} - \frac{1}{4} v^4 e^{-4it} \right) + \tau G_0 \quad (\tau = \varepsilon) \\ G_0 &= -\frac{1}{32} \left(\frac{1}{4} u^4 + 2u^3 v - 2uv^3 - \frac{1}{4} v^4 \right) \end{aligned}$$

The first-order asymptotic form of the normal form is the coefficient of t

$$\bar{H}_1 = i \left(uv + \frac{3}{16} \tau u^2 v^2 \right)$$

The first approximation G_0 for the generating Hamiltonian is the coefficient of τ , which does not depend on time.

Second approximation of the normal form. We calculate R_2

$$R_2 = H + \tau \{H - H_0, G_0\} + \frac{\tau^2}{2} \{ \{H_0, G_0\}, G_0 \}, \quad H - H_0 = \frac{i\tau}{32} (u + v)^4$$

We find

$$\{H_0, G_0\} = -\frac{i}{32} (u^4 + 4u^3 v + 4uv^3 + v^4)$$

We substitute $H - H_0$ and $\{H_0, G_0\}$ into the expression for R_2 , and then integrate $R_2(u, v)$ along trajectories of the generating system $u \rightarrow u \exp it, v \rightarrow v \exp(-it)$. The coefficient of t in the expression thus obtained is

$$\bar{H}_2 = i \left(uv + \frac{3\tau}{16} u^2 v^2 - \frac{17\tau^2}{256} u^3 v^3 \right)$$

This expression, taken at $\tau = 1$, is the normal form of the Hamiltonian of a Duffing oscillator in the second approximation. Since we are confining our attention to this approximation, the function G_1 , which is needed to construct the third approximation, will not be shown.

The next example will illustrate the fact that the proposed normalization algorithm does not require the Hamiltonian to be in polynomial form. All existing algorithms work with polynomial Hamiltonians.

Example 2. A Hamiltonian in the form of a rational function. Suppose the Hamiltonian to be normalized has the form

$$H = \frac{1}{2}(p^2 + q^2) + \frac{\epsilon}{1 + q^2}$$

The general solution of the generating system ($\epsilon = 0$) is

$$q = u \cos t + v \sin t, \quad p = -u \sin t + v \cos t$$

To construct the first approximation of the normal form, we need only integrate the Hamiltonian with respect to time, after first substituting into it the solution written out above

$$\int_0^t H(u \cos t + v \sin t, -u \sin t + v \cos t) dt = \left[\frac{1}{2}(u^2 + v^2) + \epsilon w \right] t + \epsilon w [\arctg(ws(t)) - \arctg s(t)]_0^t;$$

$$w = \frac{1}{\sqrt{1 + u^2 + v^2}}, \quad s(t) = \frac{-u \sin t + v \cos t}{u \cos t + v \sin t}$$

This yields the first approximation of the normal form (the coefficient of t)

$$\bar{H}_1 = \frac{1}{2}(u^2 + v^2) + \epsilon w$$

The first approximation of the generating Hamiltonian, which defines the canonical transformation reducing the initial Hamiltonian to this form, is the coefficient of ϵ , which is independent of t

$$G_0 = -w \left(\arctg \frac{wv}{u} - \arctg \frac{v}{u} \right)$$

The next example will illustrate the invariance of the algorithm with respect to the choice of the initial canonical variables in the Hamiltonian to be normalized. In all existing algorithms, the generating Hamiltonian must first be reduced to one of the elementary forms. In the algorithm proposed here this is not necessary.

Example 3. A Lagrange top on a vibrating base. Consider the problem of the stability of a Lagrange top when the base is vibrating along the vertical axis. The problem of the dynamics of a Lagrange top on a base vibrating in a horizontal plane was considered in [8] in connection with the phenomenon of nuclear magnetic resonance (NMR).

The vertical position of the top when in horizontal vibration becomes unstable when the vibration frequency is close to the frequency of the top's precessional oscillations. The NMR phenomenon, which is known in the literature, is a resonance of a precessional type. It is of interest to determine the conditions under which NMR of the second kind, that is, resonance of nutational type, is possible.

The Lagrangian describing the dynamics of small deviations of the top from the vertical may be written, e.g. in Krylov-Bulgakov angles

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + (h + \Delta)(x\dot{y} - \dot{x}y) + \frac{1}{2}(x^2 + y^2)(1 + 2k \cos 2ht)$$

where h is the intrinsic angular momentum, Δ defines the frequency difference, and k is the intensity of vertical vibrations of the base.

We will change to non-dimensional time $t \rightarrow t'/h$, but retain the previous notation for the new time variable in what follows. The new Lagrangian will be

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + (1 + \delta)(x\dot{y} - \dot{x}y) + \epsilon(x^2 + y^2)(1 + 2k \cos 2t), \quad \epsilon = \frac{1}{2h^2}, \quad \delta = \frac{\Delta}{h}$$

Assuming that the intrinsic angular momentum h is large, we shall assume that ϵ and δ are small parameters of the same order of magnitude.

A Legendre transformation enables one to obtain a Hamiltonian $H = H_0 + H_*$, in which the unperturbed part and the perturbation have the form

$$H_0 = \frac{1}{2}(x^2 + y^2 + p_x^2 + p_y^2) - xp_y + yp_x$$

$$H_* = -\delta(xp_y - yp_x) + (x^2 + y^2)[2\delta - \epsilon(1 + 2k \cos 2t)]$$

The relation between the generalized momenta and the initial phase variables is

$$p_x = \dot{x} - (1 + \delta)y, \quad p_y = \dot{y} + (1 + \delta)x$$

We recall that, according to the previous account, a non-autonomous Hamiltonian must be brought to autonomous form. In this case we must put $2t = q_0$ and add $2p_0$ to H_0 .

The normalization procedure for the Hamiltonian H will be implemented without diagonalizing its unperturbed part H_0 . To that end, we first write down the solution of the generating system

$$\dot{q}_0 = 2, \quad \dot{x} = p_x + y, \quad \dot{y} = p_y - x, \quad \dot{p}_x = -x + p_y, \quad \dot{p}_y = -y - p_x$$

which is

$$\begin{aligned} q_0 &= q'_0 + 2t \\ x &= x' + p'_y + (x' - p'_y)\cos 2t + (y' + p'_x)\sin 2t \\ y &= y' - p'_x - (x' - p'_y)\sin 2t + (y' + p'_x)\cos 2t \\ p_x &= -y' + p'_x - (x' - p'_y)\sin 2t + (y' + p'_x)\cos 2t \\ p_y &= x' + p'_y - (x' - p'_y)\cos 2t - (y' + p'_x)\sin 2t \end{aligned}$$

The prime indicates the values of the appropriate functions at $t = 0$.

Substituting this solution into the initial Hamiltonian, and then integrating it from 0 to t and singling out the coefficient of t , we obtain

$$\bar{H} = 2p_0 + \frac{1}{2}\xi_+ + yp_x - xp_y + \delta(y p_x - x p_y) + \frac{\delta - 2\varepsilon}{4}\xi_+ + \frac{\varepsilon k}{2}\xi_- \cos q_0 - \varepsilon k(x p_x + y p_y) \sin q_0$$

$$\xi_{\pm} = x^2 + y^2 \pm (p_x^2 + p_y^2)$$

(omitting the primes). We can now return to the original notation for the time, $q_0 = 2t$.

By the theorem formulated in Section 4, to investigate the stability of the vertical equilibrium position further, we need only consider the perturbed part of the normal form at the point $t = 0$

$$\bar{H}_* = \delta(y p_x - x p_y) + \frac{\delta - 2\varepsilon}{4}\xi_+ + \frac{\varepsilon k}{2}\xi_-$$

The characteristic polynomial of this linear system is

$$\lambda^4 + 2\gamma_+\lambda^2 + \gamma_-^2 = 0; \quad \gamma_{\pm} = \delta^2 \pm [(\delta/2 - \varepsilon)^2 - \varepsilon^2 k^2]$$

whence we obtain the stability condition $(\delta/2 - \varepsilon)^2 > \varepsilon^2 k^2$, which may be rewritten in terms of the original parameters as $(\Delta h - 1)^2 > k^2$.

6. REMARKS

1. The basic idea of the proposed algorithm was published [9] for autonomous systems in a more restricted formulation.

2. The distinctive features of the new algorithm are:

a) the sole condition defining the generating system is the existence of a known conditionally periodic general solution;

b) the criterion of the normal form used in formulating the algorithm is the condition that the vector fields of the perturbed and unperturbed parts of the system should commute;

c) the algorithm is invariant to the choice of initial phase variables in the Hamiltonian to be transformed;

d) the resonance, non-resonance, autonomous and non-autonomous cases are considered within a uniform framework;

e) the algorithm does not require the Hamiltonian to be in polynomial form;

f) the algorithm is based on the properties of the ring of asymptotic forms rather than on operations with series, this has enabled us to avoid such schemes as Newton polyhedra and to express the final result as a single recurrence formula;

g) the solution of the homological equation is obtained in the form of a quadrature.

3. Doubtful terms may appear in the process of computing the asymptotic forms. This is easily avoided if any terms of higher order than that of the asymptotic form being computed are equated to zero during the computations.

4. Reducing the algorithm to a quadrature computation does not restrict its possibilities but, on the contrary, extends them. If one assumes that the Hamiltonian to be transformed is written in polynomial form (as is indeed

done in other known algorithms), one has to integrate trigonometric harmonics in positive integer powers, which at once reduces to finite formulae.

5. In the near-resonance case, one encounters the problem of small denominators. If the spectrum of the solution of the generating system does not depend on the initial data, the case has to be converted into one of exact resonance by removing the small terms from the generating part to the perturbation. If the spectrum depends on the initial data, then, as usual, the situation is more complicated and one has to distinguish cases of the piercing of the resonance surface and those of sticking to it.

6. Further simplifications of the algorithm are possible. Using the equation of the first approximation in system (3.3), we obtain

$$\tau(H_0, G_0) = \bar{H}_1 - H_1$$

This expression may be used to reduce the number of Poisson brackets to be computed in the last term in formula (3.4):

$$\frac{\tau^k}{k!} \underbrace{\{ \dots \{ H_0, G_0 \}, G_0 \} \dots}_{i\text{-times}} = \frac{\tau^{k-1}}{k!} \underbrace{\{ \dots \{ \bar{H}_1 - H_1, G_0 \}, G_0 \} \dots}_{i-1\text{-times}}$$

For example, when constructing the second approximation, the resolvent R_2 takes the form

$$R_2 = \tau \{ \frac{1}{2} (\bar{H}_1 + H_1) - H_0, G_0 \} + H_2$$

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